

Synthetic noncommutative motives for algebraic K -theory

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Synthetic category theory is a modern axiomatic framework for higher category theory independent of set theory. In this paper, we use the language of synthetic category theory to reprove and expand upon several known results from the theory of noncommutative motives for algebraic K -theory. An additive invariant is a functor from the category of idempotent-complete stable categories which preserves filtered colimits and sends split-exact sequences to products. We give a synthetic construction of a stable presentable category of noncommutative additive motives and a universal additive invariant taking values in this category. Furthermore, we show this category of additive motives is the sifted completion of a certain small category, and we show that connective algebraic K -theory is corepresentable in this category. A localizing invariant is a functor from idempotent-complete stable categories which preserves filtered colimits and sends exact sequences to pullback squares. We show that localizing invariants valued in anima form a stable category. We give a synthetic construction of a stable presentable category of noncommutative localizing motives in which nonconnective K -theory is corepresentable, and we define a universal localizing invariant valued in this category.

Magnitude of metric spaces and spectral action models of gravity

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The spectral action and the magnitude are numerical invariants with very different origins: the first is the trace $\text{Tr}(f(D/\Lambda))$ of a cutoff applied to the Dirac operator of a spectral triple, while the second is a category-theoretic generalization of the Euler characteristic to metric spaces. For a homogeneous Riemannian manifold both have similar asymptotic expansions in which the coefficients are integrals of certain curvature forms over the manifold, suggesting a relationship not apparent from the definitions. For finite metric spaces, we construct a spectral triple recovering the metric through Connes' distance formula and show that both invariants can be written as sums over paths. For spheres we compute both exactly and expand each by Poisson summation with remainder estimates. The magnitude of the sphere of radius R is a polynomial in R up to exponentially small corrections, and the multiplicities of the positive Dirac eigenvalues on a sphere are the values of a polynomial evaluated at these eigenvalues. We show that the magnitude polynomial for the n -sphere and the multiplicity polynomial for the $(n+1)$ -sphere are proportional after rescaling R to $iR/2$. This gives a direct relation between coefficients of the magnitude and spectral action expansions for spheres. We also investigate whether a more general relationship exists for all homogeneous Riemannian manifolds.

Colored Jones tails, stability series, and the Gukov-Manolescu series

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The colored Jones polynomials of a knot are a family of Laurent polynomials associated with irreducible representations of $U_q(\mathfrak{sl}_2)$. For positive and alternating knots, the limit of the colored Jones polynomials divided by the lowest term converges to a q -series, known as the tail of the knot. Garoufalidis and Lê extended the definition of the tail to a stability series, which they proved exists for alternating knots. Based on some examples, Park proposed a possible definition for the Gukov-Manolescu series for strongly quasi-positive (SQP) knots, which was later conjectured to be equivalent to the stability series. We study the existence of the stability series for positive knots. We prove that the tail of a Murasugi sum along the canonical Seifert surfaces of positive knots is the product of their individual tails. Additionally, we disprove a conjecture by Orland et al. by identifying an SQP knot with no tail. We also study the asymptotics of some tails and find some new relations, such as between the tail of 8_{15} and a q -series studied by Garoufalidis and Zagier, which is related to the Kashaev invariant of 4_1 .

Murasugi sum and the constituent structure of strongly quasipositive knots

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Strongly quasipositive (SQP) knots are an important class of knots in low-dimensional topology and contact geometry. Each SQP knot bounds a quasipositive Seifert surface and understanding how these surfaces decompose is central to understanding the structure of SQP knots. One natural operation on Seifert surfaces is the Murasugi sum, which combines two minimal-genus Seifert surfaces into another. For SQP links, every Murasugi sum can be realized as braided plumbing of their minimal-index braided Seifert surfaces. This project investigates the Grothendieck group of strongly quasipositive links under Murasugi sum, with an emphasis on braided plumbing. A primary motivation comes from the knot invariant F_K , whose leading term is conjectured to behave multiplicatively under Murasugi sum. Identifying a collection of generating SQP links and understanding F_K on these pieces could lead to a systematic description of the invariant for all SQP links. More broadly, this work aims to clarify the algebraic and geometric structure underlying strongly quasipositive knots and links.

Density degree set of Algebraic Varieties

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We study density degree sets of algebraic varieties over number fields, extending the sweeping-curve method for products of two curves to higher-dimensional products of curves and to products with Severi–Brauer varieties.

For a product $V = C_1 \times \cdots \times C_n$ of nice curves, each with a k -rational point, we construct Zariski-dense families of iterated fiber-product curves. We obtain an explicit effective threshold, in terms of the genera and gonality of the factors, beyond which every degree belongs to $\delta(V/k)$. For products of positive-genus curves without rational-points, we establish a sharper zero-cycle-sensitive bound and deduce a uniform bound in terms of the genera and effective index.

Let C be a nice curve, let A be a central simple k -algebra of index r , and let $S = \text{SB}(A)$. For every degree- r splitting field K/k of A , we prove

$$r \delta(C_K/K) \subseteq \delta(C \times S/k) \subseteq \delta(C/k) \cap r\mathbb{N}.$$

For an elliptic curve E and nonsplit A , we further show that $r \in \delta(E \times S/k)$ if and only if E has positive Mordell–Weil rank over some degree- r splitting field of A . We also prove $\wp(C \times S/k) = \wp(C/k)$, where $\wp$ denotes the potential density degree set.

As low-degree applications, we give a branch-locus criterion implying $7 \in \delta(E \times C/k)$ for an elliptic curve E and a nice genus-2 curve C with a k -rational non-Weierstrass point. We also generalize a local obstruction criterion for non-dense quadratic points on products of genus-2 curves from $p = 3$ to arbitrary odd primes, systematically searching new explicit examples.

Recurrence and transience of excited random walks on trees and lamplighter groups

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Multi-excited random walks on a graph G are random walks in which the transition probabilities at any particular vertex x in G depend on the number of times x has been previously visited. A (usually finite) number of cookies $\omega(x, i)$ are placed on each vertex x of G , each with their own specified transition probabilities $\omega(x, e, i)$. At the i th visit of the vertex x , the random walker moves according to the transition probabilities $\omega(x, e, i)$ and consumes the cookie shortly afterward. After all cookies are consumed at a vertex, the walker moves from that vertex with transition probabilities equivalent to that of a simple random walk.

In this project, we examine multi-excited random walks on trees of exponential growth; in particular, we show that on a tree of exponential growth, there exists a recurrent multi-excited random walk with finitely many cookies on each vertex. Separately, we also look at the once-excited random walk on the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ — we provide a streamlined proof of the transience of the simple random walk on the lamplighter group, and we modify this proof to show transience of the once-excited random walk on the lamplighter group.

Mod-2 negligible cohomology of groups of order 16

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Let F be a field, G a finite group, and M an abelian group with trivial G -action. The group cohomology $H^*(G, M)$ provides information about the external structure of G . If we study the group cohomology of the absolute Galois group Γ_K of a field K , the Galois cohomology $H^*(\Gamma_K, M)$ reveals information about generalized field extensions of K by M . Relative to finite group cohomology, Galois cohomology can be difficult to compute. However, since the absolute Galois group of a field is profinite, the Galois cohomology is the colimit of all group cohomology of finite quotients of Γ_K . An element $u \in H^*(G, M)$ is negligible over F if for all field extensions K/F and all Galois extensions L/K with Galois group G , u is in the kernel of the induced homomorphism $*$: $H^*(G, M) \rightarrow H^*(\Gamma_K, M)$. Hence, the more negligible classes, the “worse” our approximation of Galois cohomology. In this project, we extend the recent computation of the mod-2 negligible classes of all groups of order 8 to groups of order 16. In particular, we compute the mod-2 negligible classes of all abelian groups of order 16 over formally real fields and fields containing sufficient roots of unity. We plan to study cohomological invariants to obtain results for the remaining abelian and non-abelian cases over a general field.

The diameter of random regular graphs

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In a graph with a fixed number of vertices, there is a trade-off between the maximum degree of any vertex and the graph’s diameter: graphs with smaller diameters invariably have vertices of higher degree. For example, the Moore bound tells us that a 3-regular graph, a graph where each vertex has degree 3, has a diameter of at least $\log_2 n$. The diameter of a random 3-regular graph is known to come close to this lower bound asymptotically, lying between $\lfloor \log_2 n \rfloor + \lfloor \log_2 \log n - \log_2 18 \rfloor + 1$ and $\log_2 n + \log_2 \log n + \log_2 6$. But what exactly the probability distribution of the diameter looks like in this range is an open question. We argue that the diameter of almost every 3-regular random graph is $\text{round}(\log_2 n + \log_2 \log n)$. Under the assumption that there are no dispensable edges (edges that do not result in the discovery of new vertices), we argue that the diameter of a standard configuration model concentrates closely around one value. We then put a tight bound on the total number of dispensable edges that can occur to demonstrate that the diameter still concentrates without this assumption.

Optimal pebbling of wedge sums of cycles and kayak paddle graphs

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In graph pebbling, pebbles are placed on the vertices of a graph. A pebbling move takes two pebbles from one vertex and places one on a neighboring vertex. The optimal pebbling number is the smallest number of pebbles in a distribution from which every vertex is reachable. We study how optimal pebbling behaves under graph composition, specifically for wedge sums of graphs constructed from cycles and paths. We establish sharp general upper and lower bounds for wedge sums and prove an edge-removal lemma reducing cycle-based pebbling problems to tree-based problems. Using this reduction and results for spider graphs, we determine the optimal pebbling numbers of octopus graphs and wedge sums of several cycles. We then study broken kayak paddle graphs, a family of trees

consisting of two branching vertices joined by a path, and derive exact formulas for their optimal pebbling numbers. Furthermore, by reducing ordinary kayak paddle graphs to broken kayak paddle graphs and optimizing over resulting cycle cuts, we obtain an exact formula for the optimal pebbling number of every kayak paddle graph.