Repeated Patterns in Arbitrary Colorings
Michael N. Manta
Mentor: David Conlon

For fixed natural numbers $n$ and $k$ and subgraph $H$, we study the fewest numbers of colors $g_k(n, H)$ such that an edge-coloring of $K_n$ does not contain $k$ vertex-disjoint color isomorphic copies of the same subgraph $H$. We also study the function $g_k^*(n, H)$, which considers the same question without the vertex-disjoint condition. Both functions are closely related to a problem posed by Conlon and Tyomkyn. Using the Local Lemma, we show that $g_k(n, H) = O(\min\{n, n^{((kv-2)/(k-1)e)}\})$ for $v = |V(H)|$, $e = |E(H)|$. With the upper bound, we show that $g_k(n, H) = \Theta(n)$ for forests. We present lower bounds of $g_k(n, C_{2t}) = \Omega(n^{1-1/t})$, $g_k(n, C_{2t+1}) = \Omega(n^{1/2})$, and $g_k(n, K_t) = \Omega(n^{1/(t-1)})$ for fixed $t$. We also prove that $g_k^*(n, H) = \Omega(n^{v/e})$ and a general lower bound for $g_k(n, H)$.

Counting n-Arcs in Projective Planes Over Finite Fields
Andrei Staicu
Mentors: Ronno Das and Elena Mantovan

Given a collection of points in the plane, classifying which subsets are collinear is a natural problem and is related to classical geometric constructions. We consider collections of points in a projective plane over a finite field such that no 3 are collinear. This is a finite set and its size is both combinatorially interesting and has deeper topological consequences. We count the number of such collections classified by the algebraic symmetries of the finite field. Variations of this problem have been considered by Glynn, Bergvall, Das, O'Connor et al. We obtain new counts for 7 points over fields of characteristic 2 and give an approach for 8 points. These new counts are governed by the existence and classification of configurations of points called the Fano plane and the Möbius-Kantor configuration.

Class Number Sums and the Prime Geodesic Theorem
Necef Alp Kavrut
Mentor: Alexander Dunn

The asymptotic behavior of the class number function for primitive binary indefinite quadratic forms is a long-standing problem in number theory. In a 1944 paper, Siegel proved Gauss' conjecture that sums of class numbers, when ordered by discriminant up to $x$ and weighted by the logarithm of their fundamental units, grew like $x^{3/2}$. Fundamental units are erratic in size however, so separating the two summands remains a tall task. In 1982, Sarnak observed that if class numbers are instead ordered by fundamental unit sizes, then analyzing sums of class numbers is equivalent to analyzing the prime geodesic theorem on the modular surface. A subsequent breakthrough paper by Iwaniec utilized Kuznietsov's sum formula to retrieve sharper bounds for the prime geodesic theorem by working on sums of Kloosterman sums and the Rankin zeta function. We investigate these two avenues for sharpening of the error term for the prime geodesic theorem.

Class Numbers of Real Cyclotomic Fields, Principal Ideals, and Regular Primes
Kenji Nakagawa
Mentor: Alex Dunn

The real cyclotomic class number problem has been studied historically through the Weber conjecture as well as through the $\mathbb{Z}_p$-extensions of Iwasawa theory. Recent work has attempted to prove the conjecture that the $\mathbb{Z}_p$-extensions all have class number one, or at least for small examples, however, the process of establishing a sufficiently small bound for the class number to conclude its exact value has been limited by proving some prime ideals are principal. We provide a proof for some known results as well as develop a theorem that gives conditions for when prime ideals are principal. These theorems avoid explicit computation which lend itself to potentially more efficiency. Furthermore, we will investigate when these theorem's conditions are satisfied as well as a possible generalization of Siegel's conjecture for regular primes.
Characterizing the Continuum-limit Behavior of Connected Components in the Barely-subcritical Erdos-Renyi Random Graph
Ali Cataltepe
Mentor: Thomas Hutchcroft

Sequences of uniform random graphs on the complete graph have been a popular object of study in combinatorics and probability theory. It is an established result that sequences with edge probabilities that decay harmonically display markedly different asymptotic behavior when the constant in their numerator is greater or less than one, with a giant component existing in the former and all components being identically distributed trees in the latter, with probability approaching 1. While combinatorial methods have been used to great success in classifying such supercritical or subcritical (respectively) sequences, the asymptotic behavior of sequences that are subcritical (resp. supercritical) while decaying slower (resp. faster) than any subcritical (resp. supercritical) harmonic sequence, which we call "barely sub/supercritical," has mostly required probability theory and couplings to better-studied stochastic processes, namely branching processes.

Of interest to us is the "continuum-limit" behavior of a barely subcritical sequence--since component sizes are identically distributed and have a tight asymptotic bound on size, we can interpret the set of components as a collection of compact metric spaces by interpreting them as a path metric space "shaped" like each component and then shrinking them by the asymptotic bound to obtain a sequence of sets of components that do not diverge in diameter. As a component’s size is the sole determinant of the distribution of shapes it can take, we can interpret each random graph in the sequence as a discrete point process on measures on measured metric spaces--the number of occurrences of a point, or measure, simply corresponds to the number of components of the size corresponding to that measure. We apply a coupling to Galton-Watson trees and the weak law of large numbers to prove that the limit of these point processes is degenerate, i.e. that the limit of any sequence of measures obtained by picking an arbitrary measure from those obtained in the point process is the Brownian Continuum Random Tree.

Stability for the Second Non-Trivial Neumann Eigenvalue of the Laplacian
Justin Toyota
Mentor: Mikhail Karpukhin

Stability for the first non-trivial Neumann eigenvalue of the Laplacian was proved for general finite-measure open in Euclidean space by Brasco and Pratelli. In this case, the shape that maximizes the first eigenvalue is the ball. It was later shown by Bucur and Henrot that the second eigenvalue is maximized among bounded open sets by the union of two disjoint balls of equal volume. In this paper, we build off this proof using rearrangement techniques found in the proof of stability for the first eigenvalue (as described by Brasco and De Philippis) to prove a stability result for the second eigenvalue.

Special Families of Faithful Metacyclic and Dicyclic Galois Covers of the Projective Line
Brian Yang
Mentor: Elena Mantovan

The study of special subvarieties of the Torelli locus has long been of great interest. We present a criterion for a dimension 0 subvariety of the Torelli locus, arising from a G-Galois cover of \( \mathbb{P} \) branched at 3 points, to be special. We develop methods to compute the complex multiplication field and type of Jacobian varieties arising from these covers. Then, we apply the formula of Shimura and Taniyama to compute the Newton polygons of these Jacobians. As an application, we consider the cases where G lies in the following two families of non-abelian finite groups: faithful metacyclic groups and dicyclic groups. We find that the Newton polygons arising from dicyclic covers are particularly well-characterized.

Ordered Product Factorizations in the Tropical Vertex Group
Melchor Herrera
Mentor: Tony Yue Yu

The general objective of this project is to use computational evidence to generate conjectures on the structure of factorizations of elements of the tropical vertex group. Relationships have been discovered between factorizations of commutators of the tropical vertex group, Euler characteristics of the moduli spaces of quiver representations, and Gromov-Witten counts of rational curves on toric surfaces, but the computational structure of the tropical vertex group itself has not yet received intense study. The factorizations are generated using a pre-existing algorithm implemented in C using the FLINT library. Further research would involve using the explicit formulas for factorizations in the tropical vertex group to yield explicit formulas for Gromov-Witten invariants.
**Morse Index of Infinite Families of Minimal Surfaces in the 3-torus**  
Miles Cua  
*Mentor: Antoine Song*

By rescaling the fundamental domain of a triply-periodic minimal surface, we construct infinite families of minimal surfaces immersed in the 3-torus. To study the asymptotic behavior of the Morse index, area, and genus for these families, we obtain rough estimates with an argument using Schoen’s curvature bound for stable minimal surfaces. We then apply the Courant nodal theorem and a domain decomposition method to obtain stronger estimates. In particular, we obtain results for the families obtained from the Schwarz $P$ surface and $D$ surface.

**Unifying Proofs of Definable Dichotomies**  
Noah D. Ortiz  
*Mentor: Zoltán Vidnyánszky*

There is a recurrent, yet not formally unified, argument in the literature of definable combinatorics about when it is possible to build a definable homomorphism from one graph $G$ to another graph $H$. By generalising to definable models with finite-arity relations and introducing a "weak projective limit", we establish sufficient conditions for the disjunction that, either $H$ can be covered by countably many sets of a certain form, or there is a continuous model morphism from $G$ to $H$. Kechris+Solecki+Todorcevic (1999) established a minimal definable graph $\mathbb{G}$ with uncountable Borel chromatic number. Lecomte (2009) and Miller (2011) presented generalisations to hypergraphs and classical arguments of $\mathbb{G}$. Carroy+Miller+Schrittesser+Vidnyánszky (2021) established a minimal definable graph of Borel chromatic number at least three with a similarly structured proof. Despite having remarkable similarities, these proofs have not been unified due to technical differences in each. Our statement covers these results in at least the finite-arity cases. By weakening from continuous to Borel, we anticipate that the statement can be generalised to relations of infinite arity.

**Extension of Shannon Entropy to Finite Categories**  
Stephanie Chen  
*Mentor: Juan Pablo Vigneaux*

Given any finite set equipped with a probability measure, one may compute its Shannon entropy or information content. The entropy is algebraically characterized by a recursive property known as the chain rule and becomes the logarithm of the cardinality of the set when the uniform probability is used. Tom Leinster introduced a generalization of cardinality for certain finite categories through the Euler characteristic. Our project aims to extend the concept of entropy to finite categories. We first generalize the category of finite probability spaces by endowing finite categories with object and transition probabilities. We experiment with notions of marginalization and conditional probability under probability-preserving functors. We explore functions defined for the finite probabilistic categories that share the recursive property of entropy and further coincide with Shannon entropy for finite sets and relate to the Euler characteristic for a "uniform" assignment of probabilities.

**Information Theoretic Trade-offs in Creating Disentangled Representations**  
Eric Paul  
*Mentor: Juan Pablo Vigneaux*

Techniques in AI such as β-VAE have shown the benefits of creating disentangled representations of data. It turns out that the use of variational disentanglement algorithms is useful even in the absence of independent generative factors. When the generative factors are dependent, we experimentally see that there is a trade-off between the two goals of creating a disentangled representation: capturing one of the generative factors and being independent of the other generative factors. In our project we reformulated the idea of disentangled representations with information theoretic terms and studied the experimentally found trade-off. We looked simple cases and found three general regimes depending on how prioritized the goal of being independent of other factors was. We then proved under certain regimes how the optimal construction is a function of the generative factors. A better understanding of the trade-off that occurs provides for a better understanding of the best disentangled representations that can be achieved and thus has implications for the potential of AI’s that depend on such representations.
Universality Theorems for Random Groups
Elia Gorokhovsky
Mentors: Melanie Wood and Omer Tamuz

In probability theory, universality is a widespread phenomenon where combining many independent variables in particular ways results in a “universal” random variable which is essentially independent of the distributions of the constituent random variables. Universality is very broadly useful because it helps us understand complex systems with lots of moving parts, despite not having complete information about smaller-scale randomness. We aim to use the same approach to understand groups, foundational mathematical structures that have been the source of open problems for decades.

Prior work (Wood, 2019) has shown that there is a form of universality for random finite abelian groups arising as cokernels of random matrices. This work has remarkable applications to number theory, topology, and combinatorics. There is a conjectural universal distribution on general random groups, which agrees with known results for abelian groups. We consider random nilpotent groups, obtaining a strict generalization of the abelian case that further supports the conjecture, and explore generalization to random non-nilpotent groups. Our future work will build on these results to establish universality theorems for general random groups.